BLACK DIAMOND SCHOOL OF ENGINEERING, JHARSUGUDA

STUDY MATERIAL

ON

STRUCTURAL MECHANICS (TH-1)

THIRD SEMESTER CIVIL ENGINEERING

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TRUSSES AND FRAMES

Trusses and frames- Trusses are structures or structural components consisting of axially loaded elements suitably connected/jointed by hinges or pins in contrast to frames which have moment resisting joints.

Common Types of Roof Trusses:-

The various types of roof trusses commonly used are as under:

(1) **Warren and Pratt Trusses:-**These trusses are commonly used for the flatter roofs for spans of roughly 12 to 38m. The warren truss is usually a little more satisfactory than the Pratt. The roofs may be completely flat for spans not exceeding 10 to 15m, but for longer spans, slopes for drainage purpose are purposely provided.

(2) **The Pitched Pratt and Howe Trusses:-** These trusses are the most common types and medium size trusses which have maximum economical spans of about 30m.

(3) **Find Truss:-**This type of truss is used for steep roofs. Fink trusses are economically used for spans upto 35m. The most of the members of fink trusses are in tension. Fink trusses are further classified as French or cambered fink and fan fink.

(4) **Bow String Trusses:-** This type of truss is used for curved roofs for spans upto 30m. These are specially suited for ware houses, supper markets, garage and small industrial buildings.

(5) **Steel Arch Truss:-**This type of truss is used for spans over 30m.

Statically Determinate and Indeterminate Trusses

The trusses where all the member forces or support reactions could be found out by using the statical equations of equilibrium alone are called determinate trusses, otherwise they are statically indeterminate (redundant) having members or reaction components in excess of that can be found out by the equations of statics alone and may require deformation/compatibility equations for solution of such type of trusses..

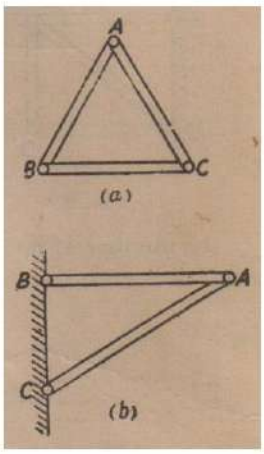
The plane struss consists of a number of bars jointed together, such that they lie in one plane and form a frame-work which is stable against any type of loading acting in the same plane. The plane trusses can be classified as :

i) Simple Trusses

ii) Compound Trusses

iii) Complex Trusses

Simple Trusses:-

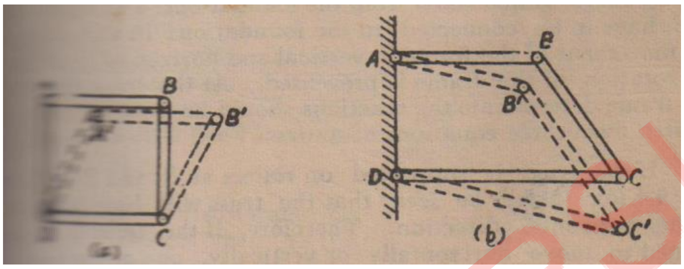


The simplest form of the truss
frame-work which should be stable
can be formed as follows:

a) By connecting three bars by means of pins to form a triangle as in Fig. (a) suitable connected to the foundations. This will form a rigid frame which will not collapse.

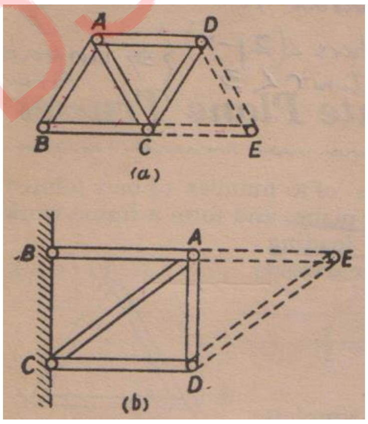
b) By taking two bars from rigid foundations and jointing them by means of a pin at the end to form a triangle as in Fig. This will form a rigid frame which will not collapse.

In any other form the frame-work will not rigid. For example, the frame-work consisting of four bars in the form of polygon ABCD, as shown in Fig. is not stable and can collapse as shown by dotted lines.



In case the frame-work consists of two bar AB and DC from the rigid foundations, their ends being connected by the third bar BC. The frame-work will collapse as shown by dotted lines in Fig.

Beginning with the rigid frame ABC by adding two bars AD and CD pinned together at D, we have rigid frame ABCD shown in Fig. The frame



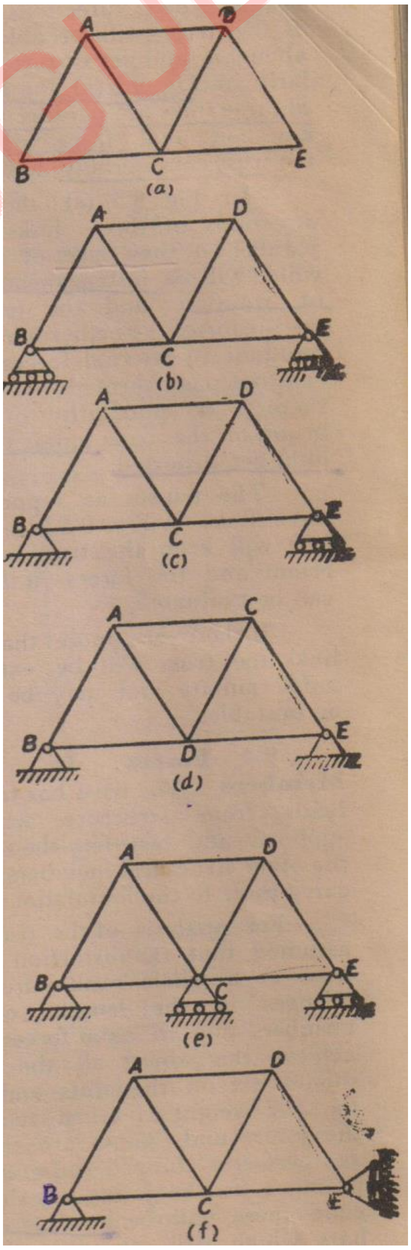
can be further extended by adding two bars which should not be in the same straight line pinned together as shown by dotted line in Fig.

There is a definite relationship between the number of bars or members m and number of joints, j in simple trusses.

In the case of the truss shown in Fig. which starts with the basic triangular frame ABC, there are three member and three joints for frame ABC and further for every joint there are two members. Leaving the basic triangle, if m is the number of members and j' is the number of joints.

In the case of the truss shown in Fig. not counting the points of attachments to the foundation as joints, there will be two members for each joint, e.i.

Connection to Foundations: The truss of Fig. will have to be connected to the foundations suitably unlike the truss in Fig. which starts from the foundation. The frame ABCED shall have to be connected to the foundations in a manner so that the movement of the frame in vertical and horizontal directions and the rotation of the frame is prevented. At the same time to make the frame determinate the reactions should be such as could be calculated from three equations of statics £V=0, £H=0 and £M = 0.

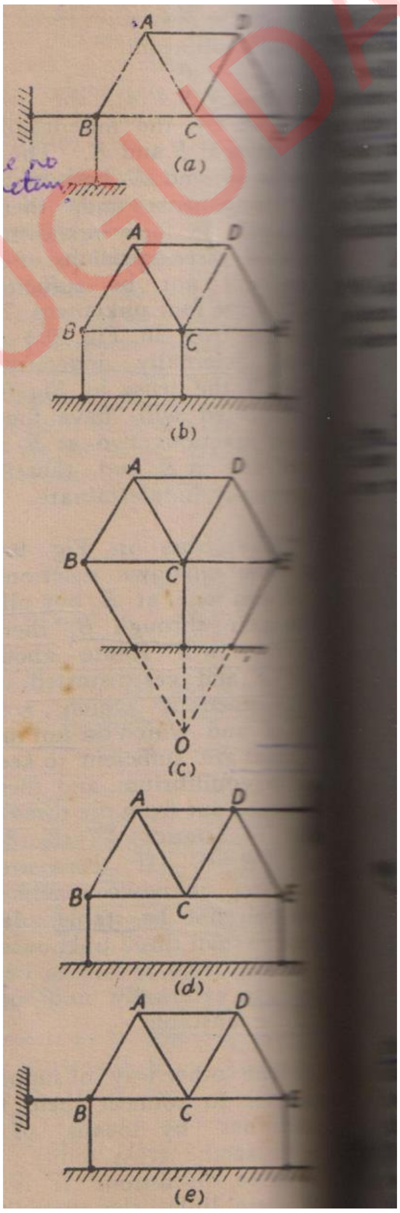


If the truss is supported on rollers at B and E as shown in Fig. it will be seen that the truss will have no constraint in the horizontal direction. Therefore, if the truss is to be constrained to move horizontally or vertically, one connection to the foundation should be a hinged one. The other connection should be such as to prevent rotation about the hinge.

The connection shown in Fig. will be adequate and will produce reactions which can be calculated, as at the hinge B, the reaction will have two components and the reaction at E will be vertical. The three unknowns can be calculated by the three equation of equilibrium.

In Fig. the truss is connected to the foundations by two hinge at B and E. The reactions at B and E both will have two components and, therefore, there will be four unknowns and the usual three equations of equilibrium will not be sufficient to evaluate the four unknowns. Therefore, the truss in Fig. is called externally indeterminate. Similarly the truss in Fig. as supported will have four unknown reactions two at B, one at C and one at E and, therefore, it is externally indeterminate.

The truss in Fig. has three unknown reactions two at B and one at E but all these three pass through B, therefore, the truss will rotate about the point B and get distorted. Thus three reactions which are not parallel and which do not meet at one point are sufficient to keep the truss in equilibrium and these can be worked out from the three equations of statics £V=0, £H=0 and £M = 0. If there are less than three unknown reactions the truss will not be stable. If there are more than three unknown reactions, the truss is said to be indeterminate, externally and may be stable or unstable.



The other way of supporting trusses is to connect them to the foundations by means of links hinged at the truss joints. From the above discussions it is clear that three links, the directions of which do not meet at any point or are not parallel, will keep the truss in equilibrium and the forces in these links can be worked out by three equations of statics.

In Fig. there links connect truss to the foundations but as their directions meet at B, the truss will be able to rotate about B and get distorted. Similarly in Fig. the directions of three links are parallel therefore, any horizontal force will distort the truss considerably.

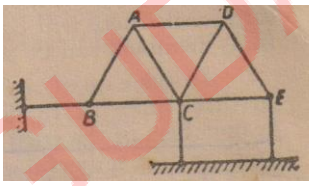
In Fig. though the directions of three links are not parallel but they meet a point 0 which will be instantaneous centre of rotation and the truss will get distorted. Furthermore if the resultant of external forces applied does not pass through the point 0, there is no possibility of equilibrium of the truss unless the three links get distorted.

The trusses as supported by three links in Fig. will keep the truss in equilibrium and the forces in the links can be evaluated.

If there are more than three links the truss will be externally indeterminate and may be stable or unstable.

Forces in Truss Members: The truss has to carry loads from structure which it supports and transfers the same to the other structural members which carry these to the foundations.

For analysis of the truss it is assumed that the distortion of the truss as a whole results from the changes in the lengths of the members due to axial forces. To achieve this object all the loads should act on the joints and even the self weight is taken to act at the joints and joints are treated as perfectly hinged. In actual practice, as the joints are slightly rigid there will be bending of the bars which will cause secondary stresses. These are neglected in the first analysis. Therefore, the members will carry only axial forces, tensile or compressive.



For analysis the truss members are assumed to be weightless, meeting at frictionless joints and the external loads are applied at the joints in the plane of the truss. If a free body diagram of each joint is drawn, it will consist of a system of co-planner forces consisting of external forces acting at the joint and axial forces induced in the bars meeting at the joint. This system of forces will be in equilibrium. The analysis of the truss is to find the internal forces induced in the bars.

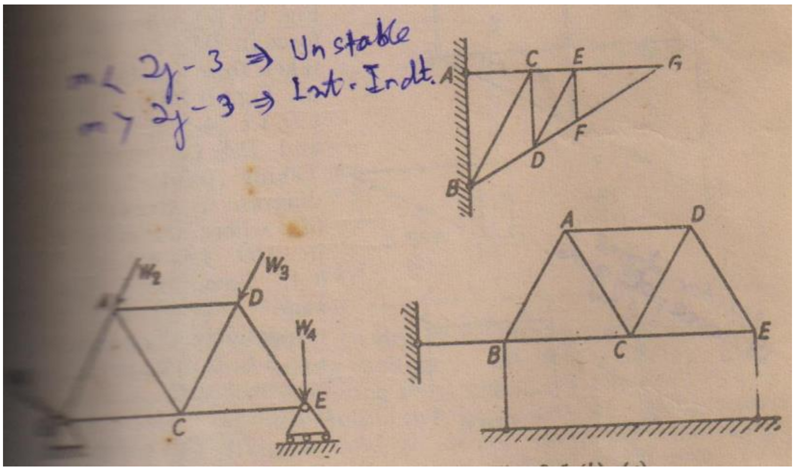
As discussed previously the rigid frame-work as shown in Fig. will satisfy the following equation between members and joints.

$m = 2j - 3$

Where m=number of members

j=number of joints

At each joint there will be two equations of equilibrium, therefore, for j joints there will be $2j$ equations. For externally determinate truss three reaction components which do not meet at a point are necessary for connecting to the foundations.



The total number of unknowns, i.e. forces in members and the three reaction components should be equal to $2j$, if the frame is to be stable and determinate. Therefore, number of members, m should be equal to $2j$-3. If the number of members is less than $2j$-3,there will be more equations than the unknown and the frame will be unstable. If the members are more than $2j$-3, the equations will not be sufficient to solve these unknowns and the frame is in the arrangement of the bras. The rigid frame formed as discussed previously will be stable and determinate.

In case of truss formed by starting with a rigid foundation as shown in Fig.9.6 (b), for a rigid truss, m=$2j$, the connection at the foundation is not counted as a joint.

At each joint there are two equations of equilibrium, therefore, for j joints there will be $2j$ equations and with these equations, forces in $2j$ members can be solved. Thus m=$2j$.

In the case of rigid frame-work connected to the foundations by three links as in Fig. if connecting links are included as members of the truss, then for j joints there are $2j$ equation and number of members necessary will be given by m=$2j$.

Analysis of Forces in Members of a Simple Truss:-The force in members of simple truss can be worked out by any of three methods:

i) Graphical

ii) Method of Joints

iii) Method of Sections

(i) Method of Joint:- In this method, an imaginary section is passed around a joint a joint in the truss, completely isolating it from the reminder of the truss. The joint becomes a free body which remains in equilibrium under the forces applied to it. For the determination of the unknown can be determined at a joint with these two equations.

(ii) Method of Sections:- In this method an imaginary section is passed completely through the truss dividing it into two free bodies, and cutting the member whose force is desired and as few other members as possible. We know that the algebraic sum of the moments of all the forces applied be a free body about any point in the plane of the truss is zero. In order to determine the force in the desired member, take moments of the force about a point so that only the desired unknown force appears in the equation. To achieve this the moments are taken about a point along the line of action of one or more of the forces of the other members.

The method of sections is very useful tool for determining the force in only one member of a truss if it is not near the end of the truss.

In writing the moment equation $\sum M = 0$, the unknown force is assumed in tension i.e. pulling away from the force body. If the solution gives a positive sign, the force is tensile and if negative the force is compressive.

Graphical method of analysis

Graphic Statics:-

Determination of reactions and forces in structural works by graphical methods, is known as graphic statics. Solution of very complicated types of trusses i.e. towers can be done very easily graphically avoiding analytical methods.

1. **Basic concepts of graphic statics:-** The following concepts may be understood clearly before attempting the analysis of trusses by graphical methods.

i.Force:- A force is represented by a vector. A vector may be drawn parallel to the force with an arrow to represent its direction and a scaled length to represent its magnitude. The vector merely represent the centres of gravity of the loads, the structural members carry.

ii.Resultant of Forces:- Two non-parallel forces intersecting at one point, may be graphically combined into one resultant with the help of a force triangle or a force parallelogram. The magnitude of the resultant force is obtained by scaling it with the same scale as that of the forces.

The resultant of three or more forces may be obtained by selecting an arbitrary point as the starting position and successive lines for each of hte forces are drawn parallel to the actual forces and scaled their proper magnitudes. The resultant of the forces is finally obtained by drawing a line from the starting point to the ending point. It may be noted that the resultant of all forces applicable to a body in equilibrium is zero and hence, the starting and closing points of polygon of forces coincide. (Fig. 15.44)

2. **Bow's Notation:-** The system of numbering the members load and reactions of a truss by placing a letter in each of the triangles of truss and in the space between each of the external loads and reactions, is known as Bow's notation.

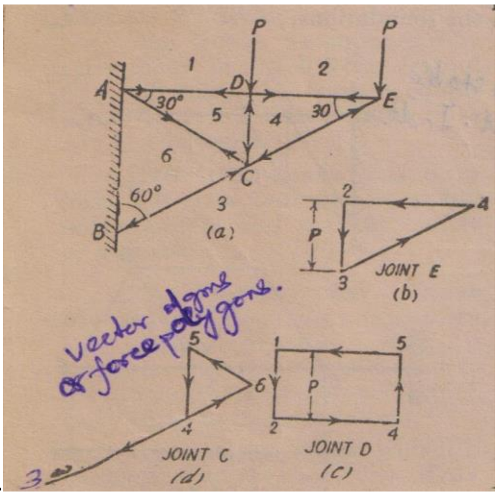
Analysis of trusses by graphical method is very much eased by adopting this mode of notation. Each external force is thus designated by a pair by letters. Similarly, the internal forces in the members of the truss are designated by the pairs of letters on each side of it.

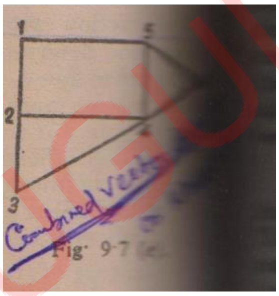
The numbering of forces is usually started from left and support and continued in clockwise direction spirally.

3. **Force polygons for individual joints:-** The resultant of forces meeting at a joint may be obtained by drawing a polygon of forces at that joint. Subsequent joints are taken out one by one and a force polygon is drawn for each.

4. **The Maxwell diagram:-** The combined diagram of the force polygons for all the joints of a truss, in which each force is represented with only one line, is called the *Maxwell diagram or Reciprocal polygon diagram*. While drawing a Maxwell diagram, the forces are considered in clockwise direction around the joints.

5.Polar diagram- It is the vector diagram of all the external forces acting on the structure represented by their arbitrary triangular components such that they have a common apex or point of resolution called pole.

In Fig. the analysis of the truss has been done by taking joint. E first and drawing the vector diagram as at Fig. the force in DE i.e. 2-4 is compressive. Taking joint D, the vector diagram is drawn in Fig. where the value of force in DE, i.e. 2-4 has been taken from the vector diagram of Fig. The member CD, i.e. 4-5 is in compression and DA i.e. 1-5 in tension. Taking joint C, the vector diagram is drawn in Fig. in CD i.e. 4-5 and CE, i.e. Fig. respectively. The member BC, i.e. 3-6 is in compression and member AC, i.e. 5-6 is in tension. 



In this construction it is sent that the vectors 2-4, 4-5, 3-4 appear in two vector diagrams. To eliminate this a single vector diagram can be drawn as shown in Fig. (e) from which forces in all the members can be found out. First vectors 1-2 and 2-3 are drawn representing external loads.

The vector diagram (2-3-4-2) represent conditions at joint E, the vector diagram (1-2-4-5-1) represents conditions at joint D, the vector diagram (5-4-3-6-5) represents conditions at joint C. To find whether the member is in compression or in tension, read the member in clockwise direction at any joint and read in the same way in the combined vector diagram, give the arrow on the member at the same joint in the directions as is read in the vector diagram. If the arrow is towards the joints, the member will be in compression if away the member will be in tension.

Analysis of frames structures:- The stresses in the various members of a framed structure may be determined by one of the following methods, discussed here under.

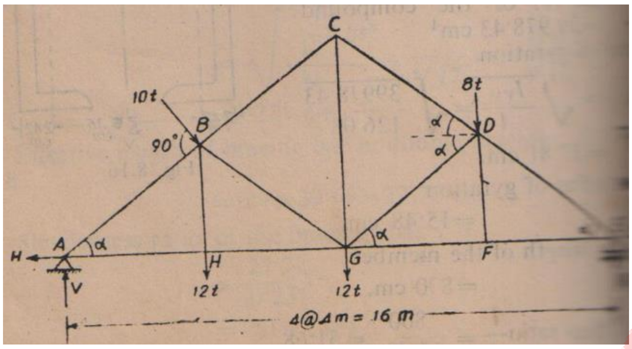
(A) Method of joints:-

Proceed as under;

1. Determine the support reactions.
2. Consider the equilibrium of the joint where forces in two members are only unknown.
3. For static equilibrium of the joint, apply the following two conditions.
4. Solve the above equations, to determine the unknown forces in the members.

The following solved examples will explain the working principal of the method joints.

Example - Find the forces in members BC, BG and HG for the given symmetrical pin jointed truss and loading as shown in Fig. 8.17, by graphical or any other method. The load of 10t acting at joint B is at right angles to the member AB and BC. The other loads act vertically downwards as shown in Fig. 8.17 ?



Solution:-

We shall find the forces in all the members analytically.

Let. R_1=vertical reaction at B $\qquad \sin\alpha = \frac{3}{4}$

V = vertical component of reaction at A $\qquad \cos\alpha = \frac{4}{5}$

H= horizontal component of reaction at A

Resolving the forces vertically, we get

$R_1+V=12+12+8+8=40$

Taking moments of the forces about A, we get

$R1=\frac{10x5+12x4+12x8+8x12}{16} = 18.125t.$

Substituting the value of R_1 in eqn. (i) we get

V=40-18.125=21.875t.

and $\quad H=10\sin\alpha = 10x\frac{3}{5} = 6t.$

Joint E:

Resolving the forces vertically,

$R_1-F_{DE} \quad \sin\alpha = 0$ assuming the force F_{DE} as compression

$F_{DE}= +18.825 \text{ x } \frac{5}{3} = 30.208t.$ (Compression)

Resolving the forces horizontally,

$F_{DE}\cos\alpha - F_{EF} = 0$ assuming the force in F_{EF} as tensile.

$F_{EF} = +30.208 + \frac{4}{5} = 24.167t.$ (Tension)

Or

Joint F:

Resolving the forces horizontally,

$F_{EF} = F_{FG} = 24.167t.$ (Tension)

Resolving the forces vertically,

$F_{FD} = 0$

Joint D:

Resolving the forces vertically,

$$-8 + F_{DF} + F_{DE}\sin\alpha - F_{DC}\sin\alpha + F_{DG}\sin\alpha = 0$$

$$-8 + 0 + 30.208x\frac{3}{5} - F_{DC}x\frac{3}{5} + F_{DG}x\frac{3}{5} = 0$$

$$-F_{DC} + F_{DG} = -16875 \quad \text{....................(i)}$$

Resolving the forces horizontally,

$$-F_{DG}\cos\alpha + F_{DE}\cos\alpha - F_{DG}\cos\alpha = 0$$

$$-F_{DC} + 30.208 - F_{DG} = 0$$

$$-F_{DC} - F_{DC} = -30.208 \quad \text{......................(ii)}$$

Solving eqns. (i) and (ii) we get

$F_{DC} = 23.54t.$ (Compression)

and $F_{DC}=6.667t.$ (Compression)

Joint C:

Resolving the forces vertically,

$$F_{CD}\sin\alpha + F_{CB}\sin\alpha - F_{CG} = 0$$

Resolving the forces horizontally,

$$F_{CD}\cos\alpha = F_{CB}\cos\alpha$$

Or $F_{CD} = F_{CB} = 23.54t$ (Compression)

Substituting the values of F_{CB} in eqn. (i)

$$2x23.54x\frac{3}{5} - F_{CG} = 0$$

$$F_{CG} = 28.25t$$

Or **Joint G:**

Resolving the forces vertically,

$$F_{CG} - 12 - F_{GB}\sin\alpha - F_{GD}\sin\alpha = 0$$

$$28.25 - 12 - F_{GB}x\frac{3}{5} - 6.667x\frac{3}{5} = 0$$

Or $F_{GB} = +12.25x\frac{5}{3} = 20.42t$ (Compression)

Resolving the forces horizontally,

$$-F_{GH} + F_{GB}\cos\alpha + F_{GF} - F_{GD}\cos\alpha = 0$$

$$-F_{GH} + 20.42X\frac{4}{5} + 24.167 - 6.667x\frac{4}{5} = 0$$

$$F_{GH} + 35.17 = 0$$

Or $\quad F_{GH} = 35.17t.$ (Tension)

Joint H:

Resolving the forces vertically,

F_{HB}=12t.

Resolving the forces horizontally,

F_{HA}=F_{HG}=35.17t.

Joint A:

Resolving the forces vertically,

$$F_{AB}\sin\alpha - V = 0$$

$$F_{AB} = \frac{V}{\sin\alpha} = 21.875x\frac{5}{3} = 36.46t$$ (Compression)

Resolving the forces horizontally,

$$F_{AB}\cos\alpha + H - F_{AH} = 0$$

$$36.46\,x\frac{4}{5} + H - 35.17 = 0$$

Or $\quad$ 29.17+6-35.17=0 $\quad$ O.K.

Member	Force	Tension / Compression
BC	33.54t	Compression
BG	20.42t	Compression
HG	35.17t	Tension

CHAPTER-2

SLOPE AND DEFLECTION OF BEAMS

Introduction

When a beam or for that matter any part of a structure is subjected to the action of applied loads, it undergoes deformation due to which the axis of the member is deflected from its original position. The deflections also occur due to temperature variations and lack-of-fit of members. Accurate values for these deflections are sought in many practical cases. The deflections of structures are important for ensuring that the designed structure is not excessively flexible. The large deformations in the structures can cause damage or cracking of non-structural elements. The computation of deflections in structures is also required for solving the statically indeterminate structures.

The deflection of beam depends on four general factors:

1. Stiffness of the material that the beam is made of,
2. Dimension of the beam,
3. Applied loads, and
4. Support conditions

Elastic curve

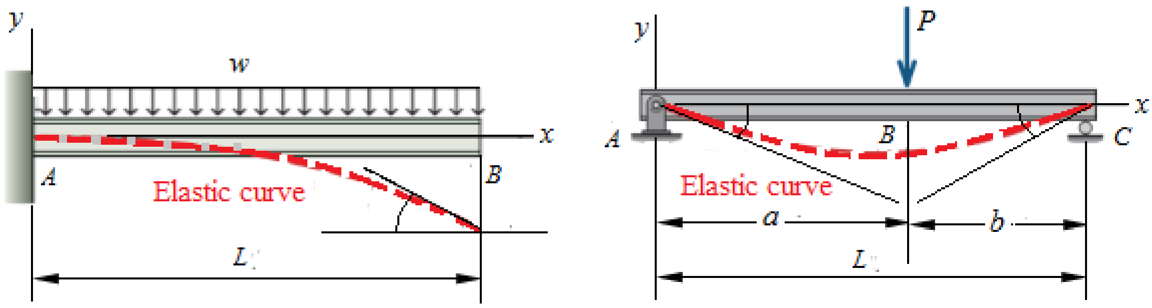
The curve that is formed by plotting the position of the neutral axis of the beam under loading along the longitudinal axis is known as the ***elastic curve***. The curve into which the axis of the beam is transformed under the given loading is called the ***elastic curve***. The nature of the elastic curve depends on the support conditions of the beam and the nature and type of loadings. The slope at a given point may be clockwise or anticlockwise measured from the original axis of the beam. Figure 1 shows the elastic curves for cantilever and simply supported beams. Sagging or positive bending moment produces an elastic curve with curvature of concave upward whereas a hogging or negative bending moment gives rise to an elastic curve with curvature of concave downward.

Deflection

The vertical displacement of a point on elastic curve of a beam with respect to the original position of the point on the longitudinal axis of the beam is called the ***deflection***.

Slope

The angular displacement or rotation of the tangent drawn at a point on the elastic curve of a beam with respect to the longitudinal axis of the original beam without loading is known as the ***slope*** at a given point.



(a) Cantilever beam (b) Simply supported beam

Figure 1

Importance of slope and deflection

Accurate values for these beam deflections are sought in many practical cases. The deflection of a beam must be limited in order to: (a) provide integrity and stability of structure or machine, (b) minimize or prevent brittle-finish materials from cracking The computation of deflections at specific points in structures is also required for analyzing a statically indeterminate structures.

Equation of elastic curve

The following assumptions are made to derive the equation of the elastic curve of a beam.

Assumptions:

1. The deflection is very small compared to the length of the beam.
2. The slope at any point is very small.
3. The beam deflection due to shearing stresses is negligible, i.e., plane sections remain plane after bending.
4. The values of *E* and *I* remain constant along the beam. If they are constant and can be expressed as functions of *x*, then the solution using the equation of elastic curve is possible.

Let us consider an elemental length *PQ* = *ds* of the elastic curve of a beam under loading as shown in the Figure 1. The tangents drawn at the points *P* and *Q* make angles $\theta$and $\theta + d\theta$ with *x*-axis. Let the coordinates of *P* and *Q* be(x, y) *and* $(x + dx, y + dy)$ respectively. The normals at *P* and *Q* meet at *C*. *C* denote the centre of curvature and ρ the radius of curvature of the part of the elastic curve between *P* and *Q*.

From the geometry of the curve, it is obvious that $ds = \rho d\theta$

or $$\rho = \frac{ds}{d\theta}$$ sass

and $$\frac{dy}{dx}=\tan\theta,\ \frac{dy}{ds}=\sin\theta,\ \text{and}\ \frac{dx}{ds}=\cos\theta$$

$$\rho=\frac{ds}{d\theta}=\frac{ds}{dx}\frac{dx}{d\theta}=\frac{\frac{ds}{dx}}{\frac{d\theta}{dx}}$$

$$\rho=\frac{\sec\theta}{\frac{d\theta}{dx}} \qquad (1)$$

Further, $$\tan\theta=\frac{dy}{dx}$$

Differentiating with respect to *x*, one can get

Asaa $$\sec^2\theta\frac{d\theta}{dx}=\frac{d^2y}{dx^2}$$

$$\frac{d\theta}{dx}=\frac{\frac{d^2y}{dx^2}}{\sec^2\theta} \qquad (2)$$

Substituting the value of $\frac{d\theta}{dx}$ in Eq.(1), one gets

$$\rho=\frac{\sec^3\theta}{\frac{d^2y}{dx^2}}$$

$$\frac{1}{\rho}=\frac{\frac{d^2y}{dx^2}}{\sec^3\theta}=\frac{\frac{d^2y}{dx^2}}{\left(\sec^2\theta\right)^{3/2}}=\frac{\frac{dy}{dx}}{\left(1+\tan^2\theta\right)^{3/2}}$$

$$\frac{1}{\rho}=\frac{\frac{d^2y}{dx^2}}{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}$$

For real life actual, the slope *dy*/*dx* is very small and its square is even smaller and hence the term $\left(\frac{dy}{dx}\right)^2$ can be neglected as compared to unit. The above expression thus becomes

$$\frac{1}{\rho}=\frac{d^2y}{dx^2} \qquad (3)$$

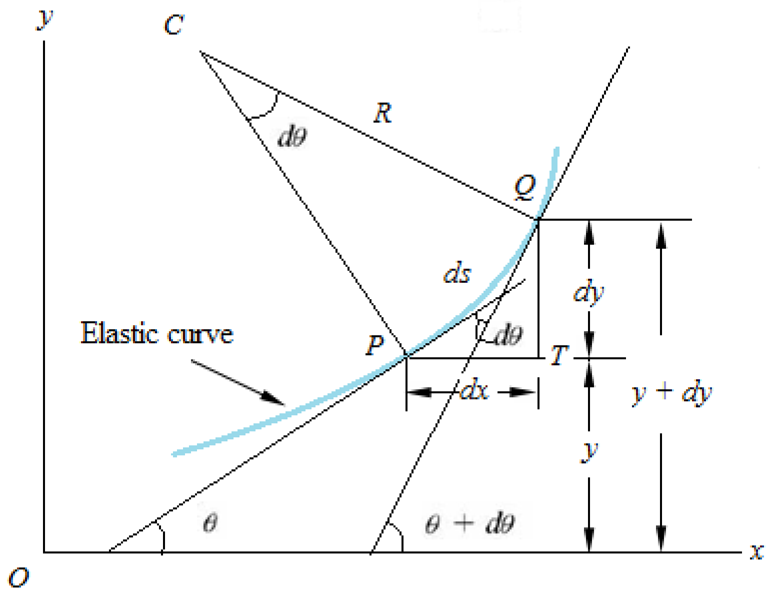


Figure 2

From theory of pure bending, it is known that

$$\frac{M}{I} = \frac{E}{\rho}$$

$$\frac{1}{\rho} = \frac{M}{EI} \tag{4}$$

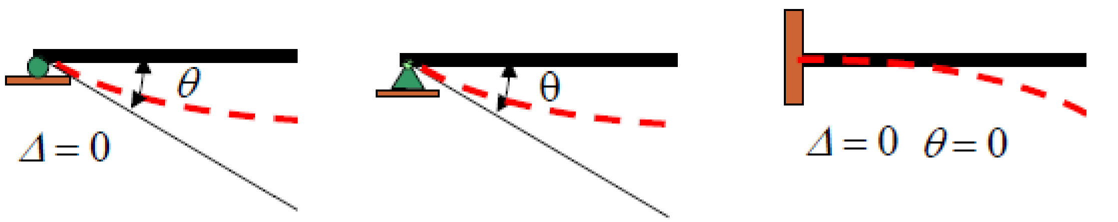
From Eq, (3) and (4) we get

$$\boxed{EI\frac{d^2y}{dx^2} = M} \tag{5}$$

Equation (5) is the governing equation of deflection of beam, also known as equation of elastic curve.

Boundary condition

The equation of elastic curve or the governing equation for deflection of the beam is a second order differential equation; hence we need to know two boundary conditions to find out two constants of integration for complete solution of the problem. The boundary conditions generally come from the support conditions, where either the slope or the deflection is known. Sometimes, due to symmetry of the beam, as in the case of a simply supported beam with point load at the centre of the beam or uniformly distributed load throughout the beam, an intermediate point representing the point of symmetry may give a boundary condition.



(a) Roller support (b) Pin support (c) Built-in or Fixed support

Figure 3

General procedure for computing deflection by integration

1. Select the interval or intervals of the beam to be used and place a set of coordinate axis on the beam with the origin at one end of an interval and then indicate the range of values of x in each interval.
2. List the variable boundary and continuity or matching conditions for each interval.
3. Express the bending moment M as a function of x for each interval selected and equate it to $EI\left(d^2y/dx^2\right)$.
4. Solve the differential equation from step 3 and evaluate all constants of integration. Calculate slope (dy/dx) and deflection (y) at the specific points.

Numerical Problems
Problem 1.
Derive the equation of elastic curve and find the slope and deflection at the free end of the cantilever beam shown in the Figure 4.

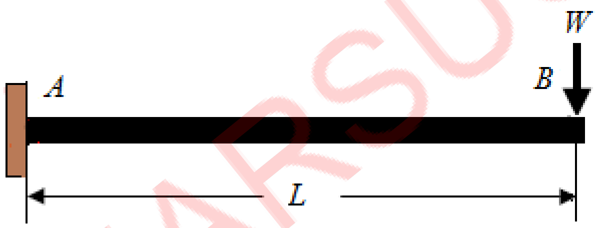


Figure 4

Solution.

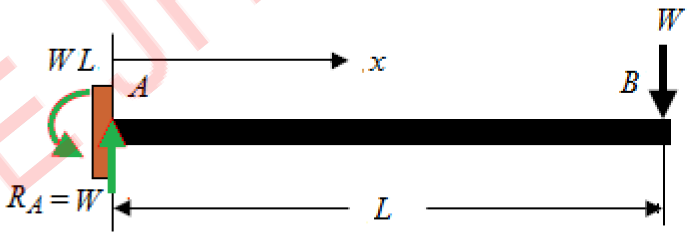


Figure 5

Determine the support reactions
Sum of the vertical forces , $\sum V = 0, \;\; R_A = W$
Sum of the vertical forces , $\sum M_A = 0, \qquad M_A = W\,L$
Taking moment about any section between A and B over the entire length of the cantilever, we have

$$M(x) = -WL + Wx$$

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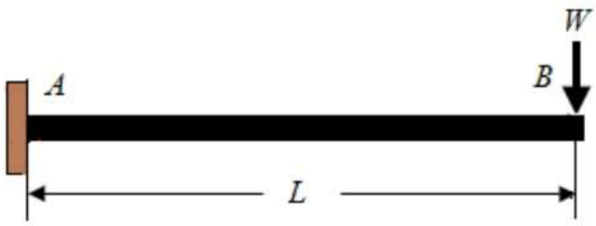
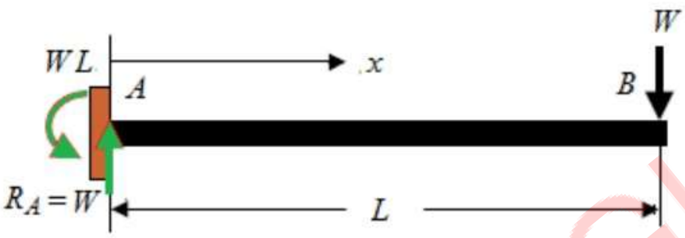


Figure 4

Solution.



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The equation of the elastic curve may be written as

$$EI\frac{d^2y}{dx^2} = -WL + Wx$$

Integrating with respect to *x*, we get

$$EI\theta = EI\frac{dy}{dx} = -WLx + \frac{Wx^2}{2} + C_1 \quad (6)$$

Integrating again with respect to *x*, we get

$$EIy = -\frac{WLx^2}{2} + \frac{Wx^3}{6} + C_1x + C_2 \quad (7)$$

The constants integration C_1 and C_2 may be determined from the boundary conditions.

$$x = 0,\ \theta = 0 \ and \ x = 0,\ y = 0$$

Substituting $x = 0,\ \theta = 0$ in Eq. ( ), we get $C_1 = 0$

Substituting $x = 0,\ y = 0$ in Eq. ( ), we get $C_2 = 0$

Substituting the values of $C_1 = 0$ and $C_2 = 0$ in Eq. ( ) and Eq. ( ), we get

General equation for slope $$EI\theta = EI\frac{dy}{dx} = -WLx + \frac{Wx^2}{2} \quad (8)$$

General equation for deflection $$EIy = -\frac{WLx^2}{2} + \frac{Wx^3}{6} \quad (9)$$

Slope at free end $(x = L)$ $$EI\theta_B = -WL^2 + \frac{WL^2}{2}$$

$$\theta_B = -\frac{WL^2}{2EI}$$

Slope at free end $(x = L)$ $$EIy_B = -\frac{WL^3}{2} + \frac{Wx^3}{6}$$

$$y_B = -\frac{WL^3}{3EI}$$

Problem 2.

A cantilever beam of length L carries a uniformly distributed load of w per unit length over its entire length. Determine the slope and deflection at the free end of the beam.

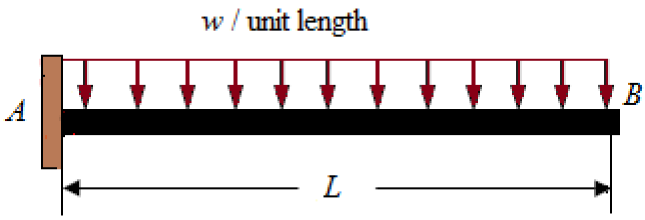

Figure 6

Solution.

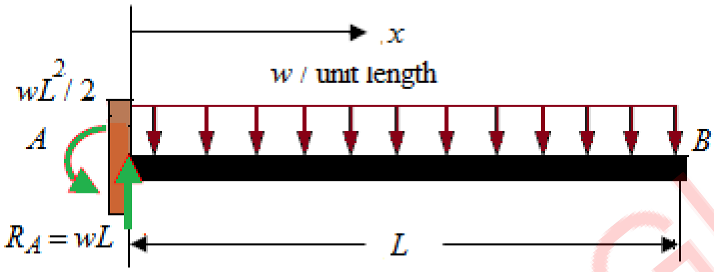

Figure 7

Determine the support reactions

Sum of the vertical forces , $\sum V = 0, \;\; R_A = wL$

Sum of the vertical forces , $\sum M_A = 0, \qquad M_A = -\dfrac{wL^2}{2}$

Taking moment about any section between A and B over the entire length of the cantilever, we have

$$M(x) = -\frac{wL^2}{2} - \frac{wx^2}{2} + wLx$$

The equation of the elastic curve may be written as

$$EI\frac{d^2y}{dx^2} = -\frac{wL^2}{2} - \frac{wx^2}{2} + wLx$$

Integrating with respect to x, we get

$$EI\theta = EI\frac{dy}{dx} = -\frac{wL^2x}{2} - \frac{wx^3}{6} + \frac{wLx^2}{2} + C_1 \qquad (10)$$

Integrating again with respect to x, we get

$$EIy = -\frac{wL^2x^2}{4} - \frac{wx^4}{24} + \frac{wLx^3}{6} + C_1x + C_2 \qquad (11)$$

The constants integration C_1 and C_2 may be determined from the boundary conditions.

$$x = 0,\; \theta = 0 \text{ and } x = 0,\; y = 0$$

The constants integration C_1 and C_2 may be determined from the boundary conditions.

$$x = 0,\ \theta = 0\ and\ x = 0,\ y = 0$$

Substituting $x = 0,\ \theta = 0$ in Eq. ( ), we get $C_1 = 0$

Substituting $x = 0,\ y = 0$ in Eq. ( ), we get $C_2 = 0$

Substituting the values of $C_1 = 0$ and $C_2 = 0$ in Eq. ( ) and Eq. ( ), we get

General equation for slope $$EI\theta = EI\frac{dy}{dx} = -\frac{wL^2x}{2} - \frac{wx^3}{6} + \frac{wLx^2}{2} \quad (12)$$

General equation for deflection $$EIy = -\frac{wL^2x^2}{4} - \frac{wx^4}{24} + \frac{wLx^3}{6} \quad (13)$$

Slope at free end $(x = L)$ $$EI\theta_B = -\frac{wL^3}{2} - \frac{wx^3}{6} + \frac{wL^3}{2}$$

$$\theta_B = -\frac{WL^3}{6EI}$$

Slope at free end $(x = L)$ $$EIy_B = -\frac{wL^4}{4} - \frac{wx^4}{24} + \frac{wL^4}{6}$$

$$y_B = -\frac{WL^3}{8EI}$$

Problem 3.
Determine the slope at the end supports and deflection at centre of a prismatic simply supported beam of length L carrying a point of W at the mid span.

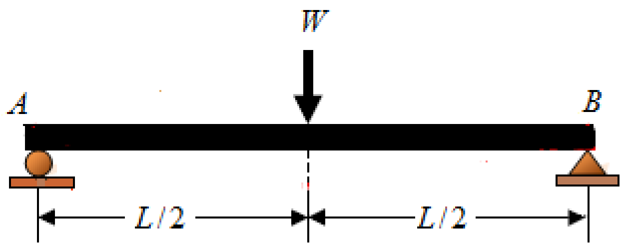

Figure 8

Solution.

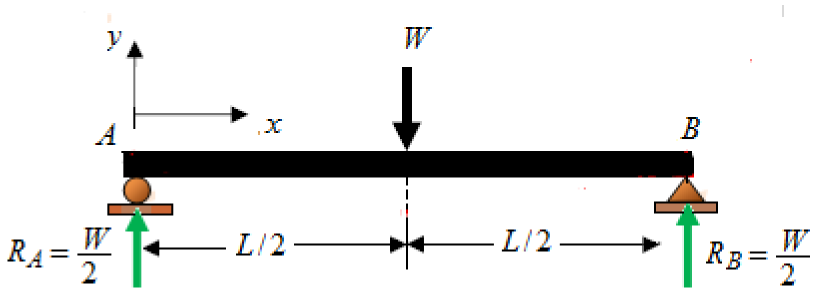

Figure 9

The beam is symmetrical, so the reactions at both ends are $\frac{W}{2}$, The bending moment equation will change beyond the centre position but because the bending will be symmetrical on each side of the centre we need to only to solve for the left hand side.

Taking moment about any section between the left hand support*A* and the centre of the beam, we have

$$M(x) = -\frac{W}{2}x$$

The equation of the elastic curve may be written as

$$EI\frac{d^2y}{dx^2} = -\frac{Wx}{2}$$

Integrating with respect to *x*, we get

$$EI\theta = EI\frac{dy}{dx} = -\frac{Wx^2}{4} + C_1 \qquad (14)$$

Integrating again with respect to *x*, we get

$$EIy = -\frac{Wx^3}{12} + C_1 x + C_2 \qquad (15)$$

The constants integration C_1 and C_2 may be determined from the boundary conditions.

At A $x = 0,\ y = 0$ (No deflection at roller supported or hinged ends)

At C $x = \frac{L}{2},\ \theta = 0$ (Tangent to the elastic curve is horizontal at the centre)

Substituting $x = \frac{L}{2},\ \theta = 0$ in Eq. (14), we get $C_1 = \frac{WL^2}{16}$

Substituting $x = 0,\ y = 0$ in Eq. (15), we get $C_2 = 0$

Substituting the values of $C_1 = \frac{WL^2}{16}$ and $C_2 = 0$ in Eq. (14) and Eq. (15), we get

General equation for slope $$EI\theta = EI\frac{dy}{dx} = -\frac{Wx^2}{4} + \frac{WL^2}{16} \quad (16)$$

General equation for deflection $$EIy = -\frac{Wx^3}{12} + \frac{WL^2x}{16} \quad (17)$$

Slope at end $A(x=0)$ $$EI\theta_A = -\frac{W(0)^2}{4} + \frac{WL^2}{16}$$

$$\theta_A = \frac{WL^2}{16EI}$$

Deflection at the centre $\left(x = \frac{L}{2}\right)$ $$EIy_C = -\frac{W}{12}\left(\frac{L}{2}\right)^3 + \frac{WL^2}{16}\left(\frac{L}{2}\right)$$

$$EIy_C = -\frac{WL^3}{96} + \frac{WL^3}{32}$$

$$y_C = -\frac{WL^3}{48EI}$$

Problem 4.

Determine the slope at the end supports and deflection at the centre of a prismatic simply supported beam shown in the Figure 10 carrying uniformly distributed load of *w* per unit length over the entire span of the beam.

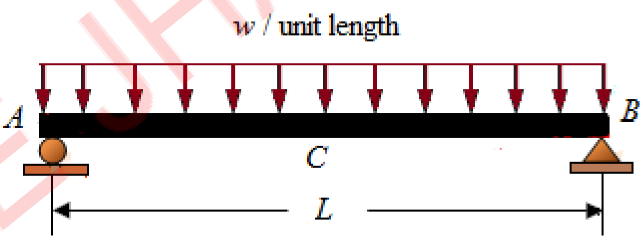

Figure 10

Solution.

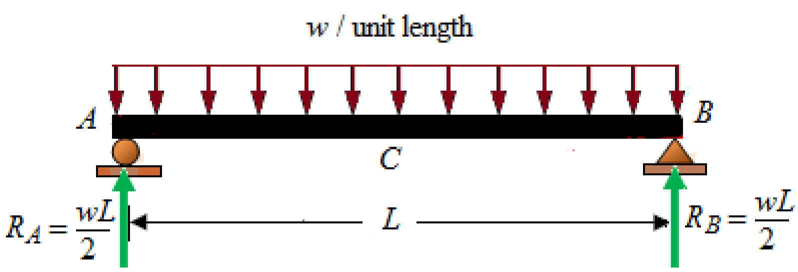

Figure 11

The beam is symmetrical, so the reactions at both ends are $\frac{wL}{2}$, The bending moment equation will change beyond the centre position but because the bending will be symmetrical on each side of the centre we need to only solve for the left hand side.

Taking moment about any section between A and B over the entire length of the cantilever, we have

$$M(x) = \frac{wLx}{2} - \frac{wx^2}{2}$$

The equation of the elastic curve may be written as

$$EI\frac{d^2y}{dx^2} = \frac{wLx}{2} - \frac{wx^2}{2}$$

Integrating with respect to x, we get

$$EI\theta = EI\frac{dy}{dx} = \frac{wLx^2}{4} - \frac{wx^3}{6} + C_1 \tag{18}$$

Integrating again with respect to x, we get

$$EIy = \frac{wLx^2}{12} - \frac{wx^4}{24} + C_1x + C_2 \tag{19}$$

The constants integration C_1 and C_2 may be determined from the boundary conditions.

At A $x = 0,\ y = 0$ (No deflection at roller supported or hinged ends)

At C $x = \frac{L}{2},\ \theta = 0$ (Tangent to the elastic curve is horizontal at the centre)

Substituting $x = \frac{L}{2},\ \theta = 0$ in Eq. (18), we get

$$EI(0) = \frac{wL}{4}\left(\frac{L}{2}\right)^2 - \frac{w}{6}\left(\frac{L}{2}\right)^3 + C_1$$

$$C_1 = -\frac{wL^3}{16} + \frac{wL^3}{48}$$

$$C_1 = -\frac{wL^3}{24}$$

Substituting $x = 0,\ y = 0$ in Eq. (15), we get $C_2 = 0$

Substituting the values of $C_1 = -\frac{wL^3}{24}$ and $C_2 = 0$ in Eq. (18) and Eq. (19), we get

General equation for slope $$EI\theta = EI\frac{dy}{dx} = \frac{wLx^2}{4} - \frac{wx^3}{6} - \frac{wL^3}{24} \quad (20)$$

General equation for deflection $$EIy = \frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3}{24}x \quad (21)$$

Slope at end $A\,(x = 0)$ $$EI\theta_A = \frac{wL}{4}(0)^2 - \frac{w}{6}(0)^3 - \frac{wL^3}{24}$$

$$\theta_A = -\frac{wL^3}{24EI}$$

Deflection at the centre $\left(x = \frac{L}{2}\right)$ $$EIy_C = \frac{wL}{12}\left(\frac{L}{2}\right)^3 - \frac{w}{24}\left(\frac{L}{2}\right)^4 - \frac{wL^3}{24}\left(\frac{L}{2}\right)$$

$$EIy_C = \frac{wL^4}{96} - \frac{wL^4}{384} - \frac{wL^4}{48}$$

$$y_C = -\frac{wL^4}{384EI}$$